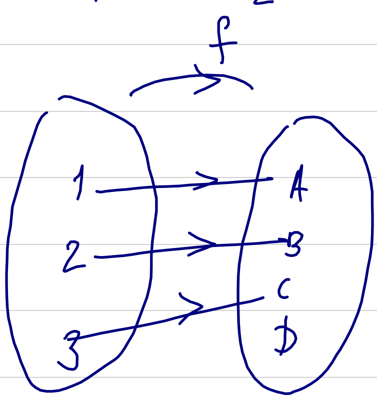
One to One functions

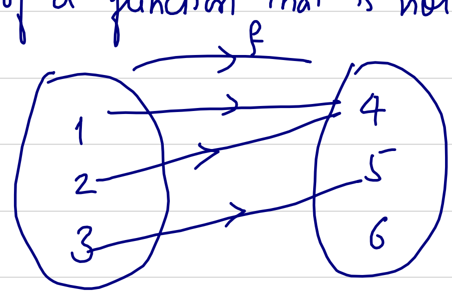
A function $f(x)$ is one-to one if no two elements in the domain map to the same element in the codomain., i.e..

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

Ex.



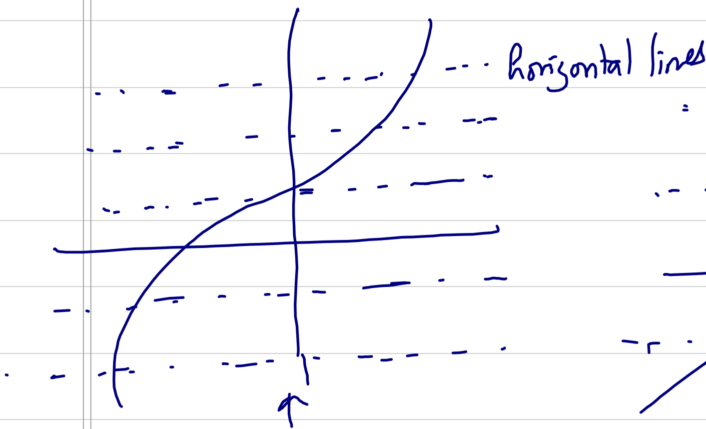
Example of a function that is not one-to-one:



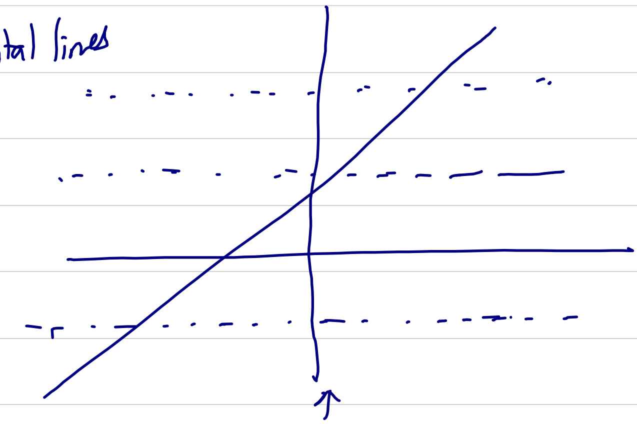
Not one-to-one because $f(1) = f(2) = 4$ and $1 \neq 2$.

Horizontal Line Test

A function is one-to-one if and only if an arbitrary horizontal line in the plane intersects the graph at most one point.

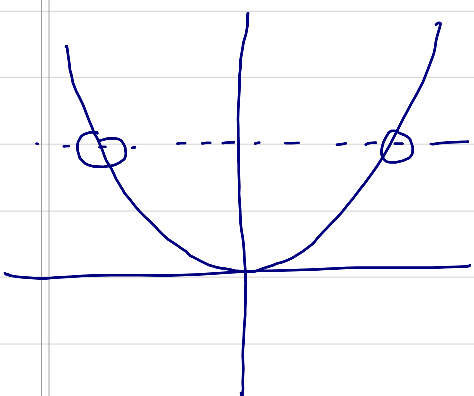


One to one

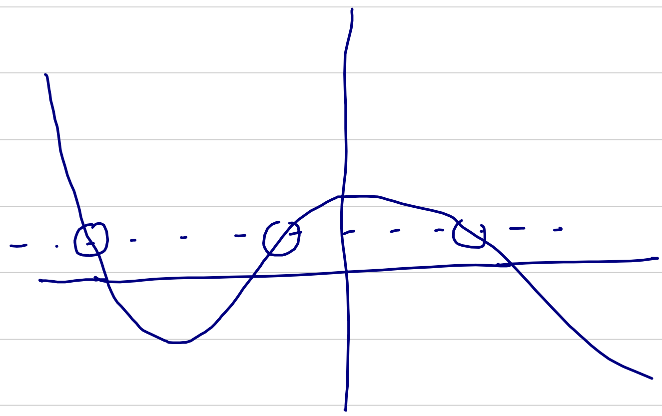


One to one

because all horizontal lines intersect at only one point.



Not one-to one

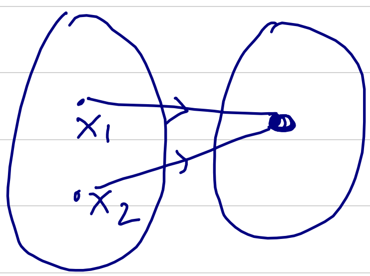


Not one-to one.

Determine whether each of the functions is a one-to-one function.

(a) $f(x) = x + 2.$

Strategy: Assume there are two numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$ unconditionally if it is one-to one.



If x_1 must equal x_2, then one to-one.

If x_1 need not equal x_2, then not one-to one.

Solution Let x_1, x_2 be two numbers such that $f(x_1) = f(x_2)$.

Then
$$f(x_1) = f(x_2)$$
$$x_1 + 2 = x_2 + 2$$
$$x_1 = x_2$$

Thus, $x_1 = x_2$. Therefore f is injective.

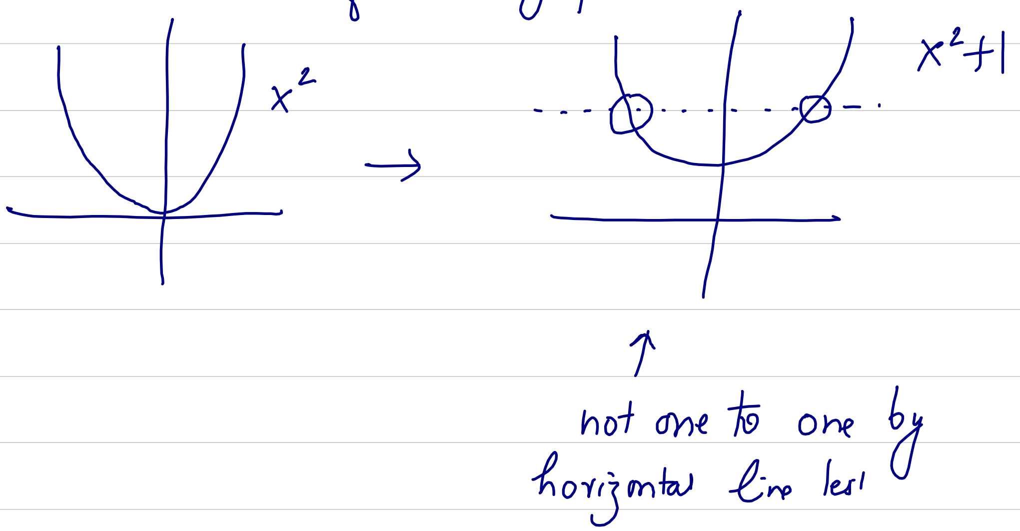
(b) $f(x) = x^2 + 1.$

Let x_1, x_2 be two numbers such that $f(x_1) = f(x_2)$,

i.e. $\quad x_1^2 + 1 = x_2^2 + 1$
$$\Rightarrow \quad x_1^2 = x_2^2$$
$$\Rightarrow \quad x_1 = \pm\sqrt{x_2^2}$$
$$\Rightarrow x_1 = \pm x_2.$$

Thus, x_1 need not equal x_2. It could equal $-x_2$.

You can also see this from the graph.



<u>Inverse Functions</u>

Let $f: X \longrightarrow Y$ be a function. For the inverse to be well defined f must be one-to-one.

$$-4x^2 + x + 1$$

$$-4x_1^2 + x_1 = -4x_2^2 + x_2$$

$$-4x_1^2 + 4x_2^2 + x_1 - x_2 = 0$$

$$-4(x_1^2 - x_2^2) + x_1 - x_2 = 0$$

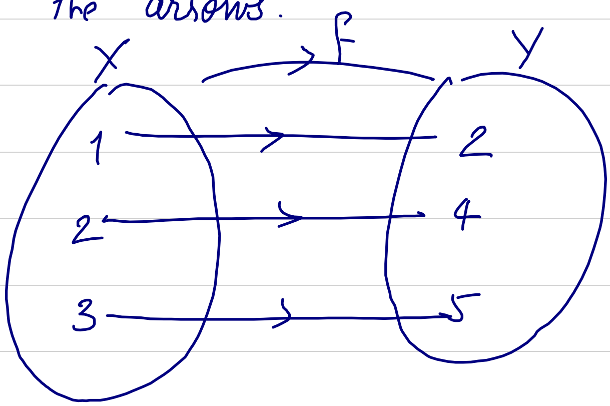
$$-4(x_1 + x_2)(x_1 - x_2) + (x_1 - x_2) = 0$$

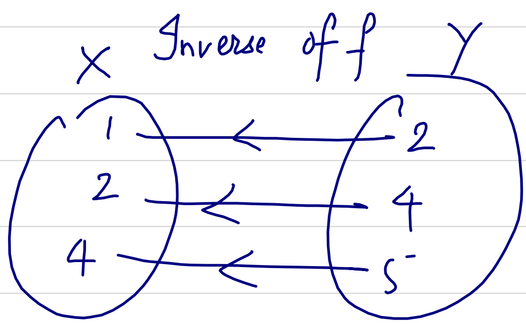
$$(x_1 - x_2)\left(-4(x_1 + x_2) + 1\right) = 0$$

$$-4x_1 - 4x_2 + 1 = 0$$

$$x_1 = \frac{1 + 4x_2}{4}$$

Assume f is one-to-one . Then the inverse of f is the function from Y to X that reverses the arrows.





In other words the inverse of f is the function $f^{-1}: Y \to X$ such that
(f inverse)

$$\begin{cases} f^{-1} \circ f(x) = x \\ f \circ f^{-1}(x) = x \end{cases}$$

$$\begin{tikzcd} x \arrow[r, "f"] & f(x) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

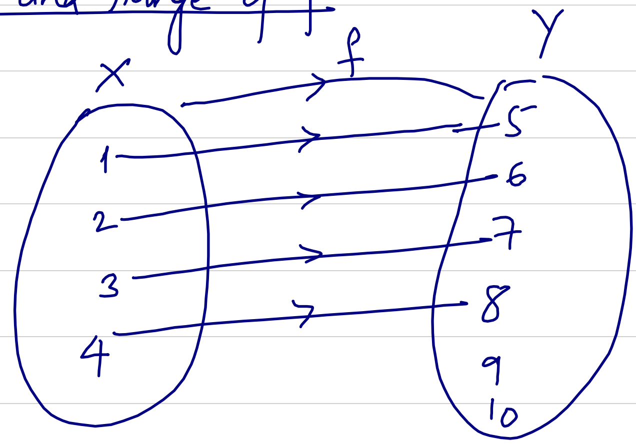
$f^{-1}(f(x)) = x$

If you apply f and then f^{-1}, You land in the same place.

$$\begin{tikzcd} f^{-1}(x) \arrow[r, bend right, "f"'] & x \arrow[l, "f^{-1}"'] \end{tikzcd}$$

$f(f^{-1}(x)) = x$

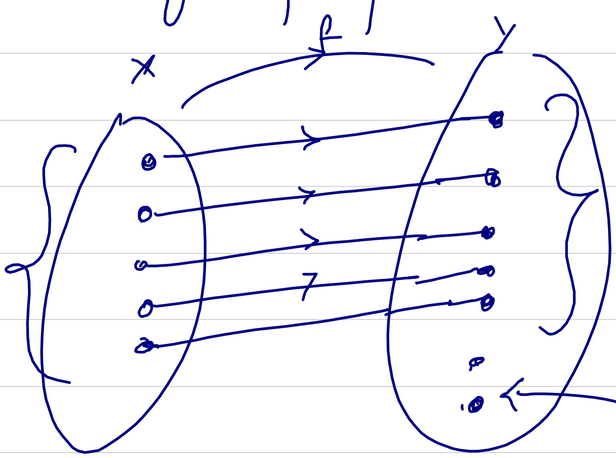
Domain and range of f^{-1}



Let f be the above function. It is clearly one-to-one.
Now f^{-1} just reverses the arrows.
But notice that Domain of f^{-1} = Range of f
Range of f^{-1} = Domain of f.

In general we have,
Domain of f^{-1} = Range of f
Range of f^{-1} = Domain of f

Domain of f = Range of f^{-1}



Range of f = Domain of f^{-1}

Why? because you cannot define f^{-1} over here

Verify that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$.

Solution. we need to veeify that

$$\begin{cases} \text{(i)} & f(f^{-1}(x)) = x \\ \text{(ii)} & f^{-1}(f(x)) = x \end{cases}$$

ATTENTION! YOU NEED TO VERIFY BOTH EQUALITIES. NOT JUST ONE.

(i) $f(f^{-1}(x))$

$$= f\left(\frac{1}{2}x - 2\right)$$

$$= 2\left(\frac{1}{2}x - 2\right) + 4$$

$$= x - 4 + 4$$

$$= x \qquad (*)$$

(ii) $f^{-1}(f(x))$

$$= f^{-1}(2x + 4)$$

$$= \frac{2x + 4}{2} - 2$$

$$= x + 2 - 2$$

$$= x \qquad (**)$$

Thus, from $(*)$ and $(**)$ we have

$$\begin{cases} f(f^{-1}(x)) = x \\ f^{-1}(f(x)) = x \end{cases}$$

Therefore, $f^{-1}(x) = \frac{x}{2} - 2$ is the inverse of f.

Verify that $f^{-1}(x) = x^2$ for $x \geq 0$, is the inverse of $f(x) = \sqrt{x}$.

Solution. We need to verify that

$$\begin{cases} f^{-1}(f(x)) = x \\ f(f^{-1}(x)) = x \end{cases}$$

We have $f^{-1}(f(x)) = f^{-1}(\sqrt{x})$

$$= (\sqrt{x})^2$$

$$= x$$

Also, $f(f^{-1}(x)) = f(x^2)$

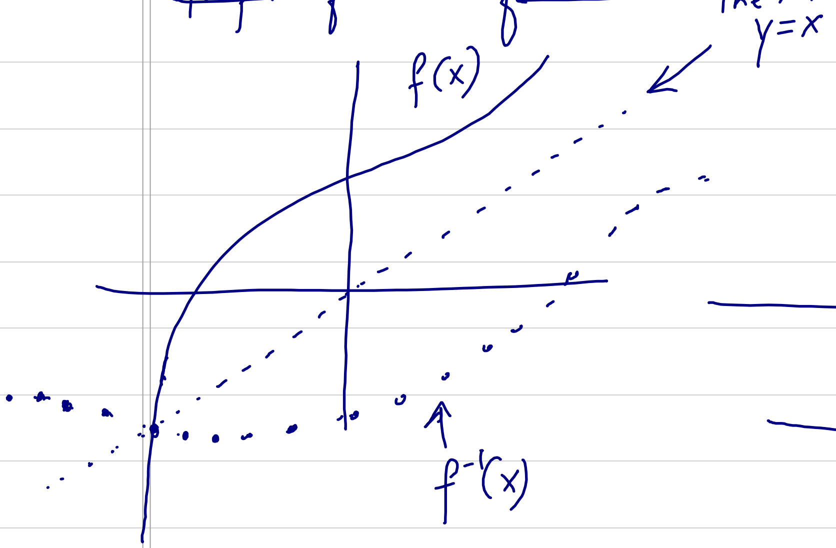
$$= \sqrt{x^2}$$

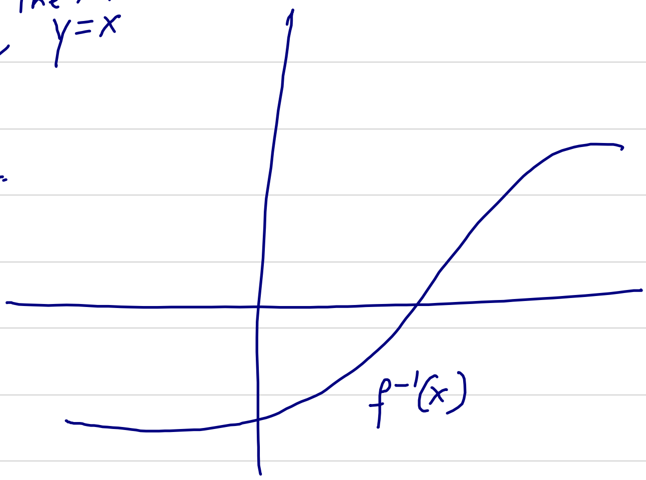
$\leftarrow$ Becareful!

$$= x \quad \text{for } x \geq 0$$

Note that $\sqrt{4} = \pm 2$, but if $x \geq 0$ which means we are looking at only the positive root, we get $\sqrt{x^2} = x$.

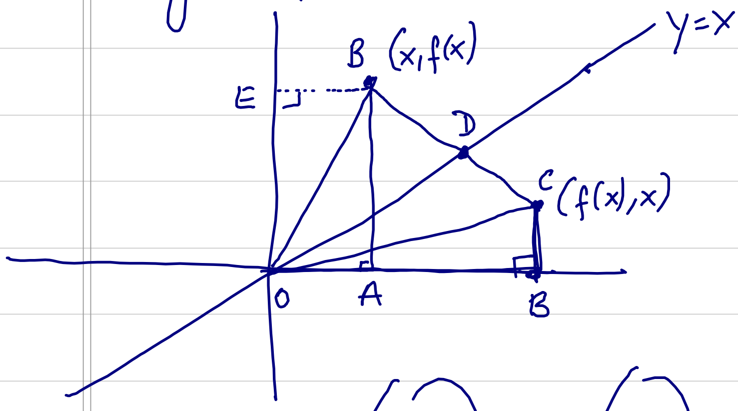
Graph of Inverse function

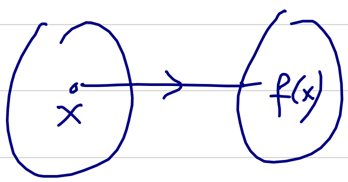




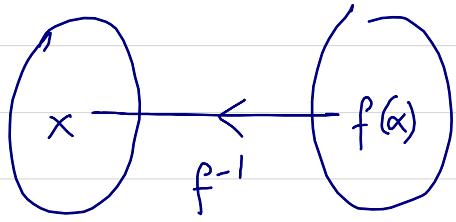
The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the line $y=x$.

Why? (This is not test material. Only if you are curious)



Say , x maps to $f(x)$. Then

$(x, f(x))$ is a point on the graph of $f(x)$. In the picture B is the point on the graph of $f(x)$.
Since the inverse f^{-1} reverses the arrows,



$f^{-1}(f(x)) = x$. Thus, $(f(x), x)$ is a point on the graph of f^{-1}. In the figure it is the point C.

We will show that the point $B = (x, f(x))$ is the reflection of the point $C = (f(x), x)$ about the line $y = x$.

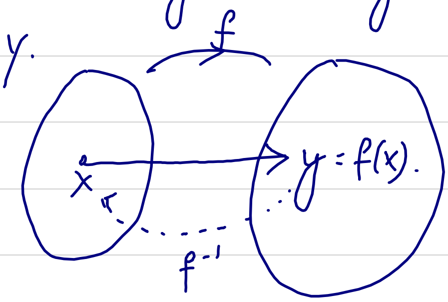
Draw the line segment $\overline{BC}$ and let it intersect the line $y=x$ at a point D. We need to show that $BC \perp OD$ (perpendicular to OD) and $\overline{BD} = \overline{DC}$.

Draw $BE \perp OE$. Note $\Delta BOE \cong \Delta COE$. Thus, $\measuredangle BOE = COB$. Since $\overline{OD}$ bisects the right angle $\measuredangle EOB$, $\measuredangle EOD = \measuredangle DOB = 45°$. Thus by subtracting $\measuredangle BOE$, $\measuredangle COB$ from $\measuredangle EOD$, $\measuredangle DOB$ respectively, $\measuredangle BOD = \measuredangle DOC$.

Note $\overline{OB} = \overline{OC}$. Hence ΔOCB is isoceles. Since OD bisects $\measuredangle BOC$, $OD \perp BC$ and $\overline{BD} = \overline{DC}$. ∎

Finding inverse of functions

Let y be in the range and say that $f(x) = y$, i.e., x maps to y.



To find the inverse of f we need a formula to find x from y.

Ex.

Find inverse of $f(x) = -3x + 5$

Solution Let y be in the range of f. Let $f(x) = y$.

So

$$-3x + 5 = y$$

$$\Rightarrow \quad -3x = y - 5$$

$$\Rightarrow \quad x = \frac{y-5}{-3}$$

[We want to get x from y. Now we want to write x as some function of y]

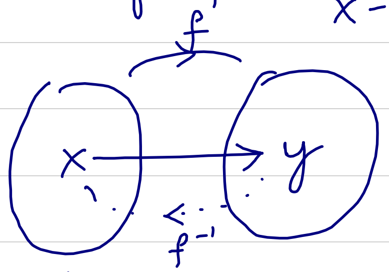
$$\therefore \quad f^{-1}(y) = \frac{y-5}{3}.$$

Exercise

Find the inverse of $f(x) = \sqrt{x+2}$

Find the inverse of $f(x) = \dfrac{4}{x-1}$, $x \neq 1$.

Sol.



Let y be in the range and say that $f(x) = y$. So

$$\frac{4}{x-1} = y$$

($x = \underbrace{?}_{\text{terms involving only } y}$. In other words we need to isolate x)

$$\frac{4}{y} = x - 1$$

or, $x = \dfrac{4}{y} + 1$

$\therefore \quad f^{-1}(y) = \dfrac{4}{y} + 1$

Given f find its inverse. State the domain and range of f and f^{-1}.

$$f(x) = \sqrt{\frac{x+1}{x+2}}$$

You already know how to find domains. I should have mentioned the technique for find range earlier. But here it is.

Let y be in the range of f. Then $y = f(x)$ for some x., i.e.. $f(x) = y$

$$\sqrt{\frac{x+1}{x+2}} = y$$

The techique is the same as for finding inverses. You need to solve for x.

$$\frac{x+1}{x+2} = y^2$$

$$x+1 = (x+2)y^2$$

$$x+1 = xy^2 + 2y^2$$

$$x - xy^2 = 2y^2 - 1$$

$$x(1-y^2) = 2y^2 - 1$$

$$x = \frac{2y^2-1}{1-y^2}$$

Now follow the technique for domains. What are the possible values of y? We know $1-y^2$ cannot be zero. If $1-y^2=0$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = \pm 1.$$

Therefore Range of f is everything except 1 and -1.
Range of $f = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Domain of f.

$$f(x) = \sqrt{\frac{x+1}{x+2}}$$

There is a square root and there is a denominator.

So. (i) $\frac{x+1}{x+2} \geq 0$

(ii) $x+2 \neq 0 \Rightarrow x \neq -2$

From (i) $\frac{x+1}{x+2} \geq 0$

when is a fraction ≥ 0. when (i') both $x+1 \geq 0$ & $x+2 > 0$

(ii') both $x+1 \leq 0$ & $x+2 < 0$

From (i') $x+1 \geq 0$ & $x+2 > 0$

$\Rightarrow x \geq -1$ $\Rightarrow x > -2$

$\therefore x \in [-1, \infty)$

From (ii') $x+1 \leq 0$ & $x+2 < 0$

$\Rightarrow x \leq -1$ $\Rightarrow x < -2$

$\therefore x \in (-\infty, -2)$

From (i') & (ii'), $x \in (-\infty, -2) \cup (-1, \infty)$

$\therefore$ Domain is $(-\infty, -2) \cup (-1, \infty)$ (2 is not included)

Inverse of f Let y be in the range of f.

Then $y = f(x)$

$$\therefore y = \sqrt{\frac{x+1}{x+2}}$$

$$\Rightarrow y^2 = \frac{x+1}{x+2}$$

$$\Rightarrow x+1 = (x+2)y^2$$
$$\Rightarrow x+1 = xy^2 + 2y^2$$
$$\Rightarrow x - xy^2 = 2y^2 - 1$$
$$\Rightarrow x(1-y^2) = 2y^2 - 1$$
$$\Rightarrow x = \frac{2y^2-1}{1-y^2}$$

$$\therefore f^{-1}(y) = \frac{2y^2-1}{1-y^2}.$$

<u>Domain of f^{-1}</u>:

Recall that domain of f^{-1} = Range of f.
$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Recall that range of f^{-1} = domain of f
$= [-1, \infty)$.